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SUMMARY

The velocity potential, lift force, moment, and propulsive force on a two-dimensional airfoil in a stream of periodically varying angle of attack have been derived on the basis of nonstationary incompressible potential-flow theory which includes the effect of the continuous sheet of vortices shed from the trailing edge. Application of these results was made in an analysis of the variation with frequency of the propulsive force on an airfoil in an oscillating stream and in an analysis of the problem of forced vibrations of an airfoil in an oscillating stream with consideration of the stiffness of the airfoil and the position of its torsion axis. It was shown that when the torsion axis of the airfoil is ahead of the quarter-chord point the amplitude of vibrations is generally not large, but when the torsion axis is behind the quarter-chord point certain conditions exist under which dangerous amplitudes of vibration may occur. The nonuniform response which was found for a freely hinged airfoil restricts the use of such a device as a flow-measuring instrument to the measurement of only very low-frequency angular variations in an oscillating stream.

It is expected that the results of the theoretical treatment of the propulsive force will be useful in considerations of counterrotating-propeller efficiencies and that the analysis of the problem of forced vibrations will be useful in design considerations of lifting surfaces operating in oscillating streams; for example, wind-tunnel fan blades behind a set of prerotation vanes, or tail surfaces in fluctuating wakes.

INTRODUCTION

The phenomenon of an airfoil in an oscillating stream (that is, a stream of which the angle of attack varies periodically) is encountered in many phases of aeronautics. For example, the effect of a set of prerotation vanes upon a wind-tunnel fan blade is to produce periodic disturbances through which the fan blades pass. Further, a consideration of counterrotating propellers shows that the rearward blades operating in the helical wake produced by the forward blades are in a stream of varying angle of attack. As another example, the horizontal tail of an airplane may be subjected to forces induced by fluctuations in the angle of the wing wake.

The theory of nonstationary motion around airfoils with consideration of partial motions of the fluid has been developed by Garrick (reference 1), Küssner (reference 2), and others. The present alternative treatment leads to a value for the lift force on an airfoil in an oscillating stream which is in agreement with that obtained in references 1 and 2. The treatment in reference 2 leads to general results for the propulsive force. The present paper, employing somewhat different derivations, gives explicit results for the propulsion as well as for the lift on an airfoil in an oscillating stream.

Two problems which arise in cases of oscillating flows are (1) the production of vibrations and (2) the so-called "Katzmayr effect" (reference 3) or existence of a propulsive force. With regard to problem (1) the object of the present paper is to examine theoretically the dynamics of an airfoil in an oscillating stream and to determine under what conditions dangerous amplitudes of vibration may occur. A special case for which the torsional stiffness is zero is treated with a view to the possibility of using a small freely hinged airfoil as a device for measuring the angular amplitude of an oscillating stream. Previous work on this problem (reference 4) has been done for the case in which the stiffness of the airfoil was expected to give large vibrations. With regard to problem (2), which is of importance in considerations of counterrotating-propeller efficiencies, a theoretical investigation is made of the horizontal forces experienced by an airfoil in an oscillating stream.

The theoretical development is divided into three parts: (1) derivation of the lift force and moment acting on an airfoil in an oscillating stream, (2) derivation of the propulsive force, and (3) derivation and solution of the equation of motion of an airfoil executing torsional vibrations in an oscillating stream. The theoretical methods used in the derivation of the lift forces and moments consist in an extension of the methods of Theodorsen (reference 5). For the derivation of the propulsive force application is made of the method outlined by von Kármán and Burgers (reference 6 pp. 52 and 306). The following usual assumptions are made throughout: (a) incompressible potential flow,

(b) two-dimensional flat-plate airfoil, (c) small oscillations, and (d) plane wake extending from trailing edge to infinity.

SYMBOLS

b	half chord of airfoil
a	x-coordinate of torsion axis of airfoil
α	angle of attack of airfoil measured clockwise from horizontal
β	angle of airstream from mean direction measured positive counterclockwise
x	horizontal coordinate; nondimensional with respect to b
t	time
v	stream velocity
ν	frequency $\left(\frac{\omega}{2\pi}\right)$
ω	circular frequency
k	reduced frequency $\left(\frac{\omega b}{v}\right)$
$C(k)$	Theodorsen's C-function from reference 5 $(F + iG)$
F,G	real and imaginary parts of C-function
p	local static pressure
ρ	air density
P	perpendicular force
M_a	pitching moment about $x = a$ measured positive counterclockwise
I_a	moment of inertia about $x = a$
R_a	torsional stiffness of wing
s	propulsive force

φ	noncirculatory velocity potential
φ_Γ	circulatory velocity potential
U	strength of wake discontinuity
ψ	phase angle
$J_0(k),J_1(k)$	Bessel functions of the first kind and zero and first order
$Y_0(k),Y_1(k)$	Bessel function of the second kind and zero and first order
γ	strength of vorticity distribution on the airfoil
m	mass per unit length of wing
r_a	radius of gyration divided by b $\left(\sqrt{\frac{I_a}{mb^2}}\right)$
κ	ratio of mass of cylinder of air of diameter equal to chord of wing to mass of wing $\left(\frac{\pi\rho b^2}{m}\right)$
C	constant

LIFT FORCE AND PITCHING MOMENT

In accordance with Theodorsen (reference 5), the forces due to the noncirculatory flow and to the effect of the wake are treated separately.

Noncirculatory force and moment.- Consider an airfoil of chord 2b at zero angle of attack with respect to the average direction of a sinusoidal stream traveling to the right (fig. 1). If the amplitude β_0 of angle-of-attack change in the stream is small then the horizontal and vertical components of the velocity, respectively, are given by

$$v_L = v \cos \beta \approx v$$

and

$$v_P = v \sin \beta \approx v\beta$$

where, with the assumption of sinusoidal oscillations,

$$\beta = \beta_0 e^{i(\omega t - kx)}$$

Thus, the airfoil may be considered as being in a uniform horizontal stream of velocity v plus a vertical sinusoidal gust of the form

$$\epsilon(x) = v\beta_0 e^{i(\omega t - kx)}$$

The velocity potential φ for such a normal-velocity distribution is (appendix A)

$$\varphi = -\frac{bv\beta_0}{k}\sqrt{1 - x^2}\, e^{i(\omega t - kx)} \int_0^k e^{iux} J_0(u)\, du \tag{1}$$

where $J_0(u)$ is a Bessel function of the first kind and zero order.

Use of the equation of motion for nonstationary flow gives

$$\frac{\partial w}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}w^2\right)$$

where

w fluid velocity

p local static pressure

ρ air density

and the substitution $w = v + \frac{\partial\varphi}{\partial x}$ gives the pressure difference Δp at the point x as

$$\Delta p = -2\rho\left(\frac{v}{b}\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right) \tag{2}$$

Integration of this local pressure difference over the length of the airfoil gives as the total force P (see appendix B, equation (B8))

$$P = 2\pi i\rho b v^2 \beta_0 J_1(k) e^{i\omega t} \tag{3}$$

where $J_1(k)$ is a Bessel function of the first kind and first order.

The noncirculatory moment about $x = a$ (fig. 2) is obtained from the integral

$$M_a = b^2 \int_{-1}^{1} \Delta p(x - a)\, dx$$

which yields (appendix B, equation (B16))

$$M_a = -\pi\rho b^2 v^2 \beta_0 e^{i\omega t}\left[2iaJ_1(k) + J_0(k)\right] \tag{4}$$

Circulatory force and moment.- The velocity potential of an element of vorticity $-\Delta\Gamma$ at a position x_0 in the wake and its mate $\Delta\Gamma$ distributed over the airfoil is (reference 5)

$$\varphi_{xx_0} = -\frac{\Delta\Gamma}{2\pi}\tan^{-1}\frac{\sqrt{1 - x^2}\sqrt{x_0^2 - 1}}{1 - xx_0} \tag{5}$$

The element $-\Delta\Gamma$ moves to the right relative to the airfoil with a velocity v. Thus,

$$\frac{\partial \varphi_{xx_0}}{\partial t} = v \frac{\partial \varphi_{xx_0}}{\partial x_0}$$

Substituting this expression and $\dfrac{\partial \varphi_{xx_0}}{\partial x}$ into equation (2) and integrating the effect of the entire wake on the airfoil yields the force

$$P = -\rho v b \int_1^\infty \frac{x_0}{\sqrt{x_0^2 - 1}} U \, dx_0 \tag{6}$$

where $U\,dx_0$ is the element of vorticity $\Delta\Gamma$ at the point x_0.

The Kutta condition requires that at the trailing edge of the plate the induced velocity equals zero; therefore, at $x = 1$

$$\left[\frac{\partial}{\partial x} \left(\varphi_\Gamma + \varphi\right)\right]_{x=1} = 0$$

where

$$\varphi_\Gamma = b \int_1^\infty \varphi_{xx_0} \, dx_0$$

Introducing the potential φ from equation (1) results in

$$\frac{1}{2\pi} \int_1^\infty \sqrt{\frac{x_0 + 1}{x_0 - 1}} U \, dx_0 = -\frac{\nu\beta_0}{k} e^{i(\omega t - k)} \int_0^k e^{iu} J_0(u) \, du \tag{7}$$

Let

$$Q = -\frac{v\beta_0}{k} e^{i(\omega t - k)} \int_0^k e^{iu} J_0(u)\, du \tag{8}$$

where u is an integrating variable. Equation (8) becomes (appendix C)

$$Q = -v\beta_0 \left[J_0(k) - iJ_1(k)\right] e^{i\omega t} \tag{9}$$

Combining equations (6), (7), and (8) and assuming the wake to be of the form

$$U = U_0 e^{i(\omega t - kx_0)}$$

gives for the circulatory force

$$P = -2\pi\rho v b Q \frac{\int_1^{\infty} \frac{x_0}{\sqrt{x_0^2 - 1}} e^{-ikx_0}\, dx_0}{\int_1^{\infty} \frac{x_0 + 1}{\sqrt{x_0^2 - 1}} e^{-ikx_0}\, dx_0} \tag{10}$$

Similarly, the circulatory moment which is obtained from

$$M_a = b^2 \int_{-1}^{1} \Delta p(x - a)\, dx$$

is (reference 5)

$$M_a = -2\pi\rho v b^2 Q \left[\frac{1}{2} - \left(a + \frac{1}{2}\right) \frac{\int_1^{\infty} \frac{x_0}{\sqrt{x_0^2 - 1}} e^{-ikx_0}\, dx_0}{\int_1^{\infty} \frac{x_0 + 1}{\sqrt{x_0^2 - 1}} e^{-ikx_0}\, dx_0}\right] \tag{11}$$

The integral expression in equations (10) and (11) is Theodorsen's C-function (reference 5). Thus,

$$P = 2\pi\rho b v^2 \beta_0 C(k)\left[J_0(k) - iJ_1(k)\right] e^{i\omega t} \tag{12}$$

and

$$M_a = 2\pi\rho v^2 b^2 \beta_0 \left[\frac{1}{2} - \left(a + \frac{1}{2}\right)C(k)\right]\left[J_0(k) - iJ_1(k)\right]e^{i\omega t} \tag{13}$$

Adding equations (3) and (12) gives for the total force

$$P = 2\pi\rho b v^2 \beta_0 \left\{C(k)\left[J_0(k) - iJ_1(k)\right] + iJ_1(k)\right\} e^{i\omega t} \tag{14}$$

This expression agrees with that given by Garrick (reference 1) and Küssner (reference 2) in which somewhat different methods of derivation are used from those used in the present paper.

Adding equations (4) and (13) gives, for the total moment about $x = a$,

$$M_a = -\pi\rho b^2 v^2 \beta_0 \left[2iaJ_1(k) + J_0(k)\right]e^{i\omega t}$$

$$+ 2\pi\rho b^2 v^2 \beta_0 \left[\frac{1}{2} - \left(a + \frac{1}{2}\right)C(k)\right]\left[J_0(k) - iJ_1(k)\right]e^{i\omega t} \tag{15}$$

Examination of equations (14) and (15) leads to

$$M_a = -b\left(a + \frac{1}{2}\right)P \quad .$$

This equation means that the center of pressure is at the quarter-chord point of the airfoil.

PROPULSIVE FORCE

According to von Kármán and Burgers (reference 6, p. 52), if the strength of the vorticity distribution at the leading edge of the plate is of the form

$$(\gamma)_{x=-1} = \left(\frac{2C}{\sqrt{b}\sqrt{x+1}}\right)_{x=-1} \tag{16}$$

then the suction or propulsive force acting on the airfoil s has the value

$$s = \pi\rho C^2 \tag{17}$$

where C is a constant. The vortex strength is given by the sum of the tangential velocities on the two sides. Thus

$$\gamma = \frac{2}{b}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi_\Gamma}{\partial x}\right)$$

and, therefore,

$$C = \left[\frac{\sqrt{1+x}}{\sqrt{b}}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi_\Gamma}{\partial x}\right)\right]_{x=-1} \tag{18}$$

Carrying out the indicated calculations in appendix D leads to

$$C = -\sqrt{\frac{b}{2}}\, v\beta_0 e^{i\omega t}\left\{\left(J_0 + iJ_1\right) + \left(J_0 - iJ_1\right)\left[\frac{J_1 - Y_0 - i\left(J_0 + Y_1\right)}{J_1 + Y_0 + i\left(J_0 - Y_1\right)}\right]\right\}$$

where J_0 means $J_0(k)$ and so forth and $Y_0(k)$ and $Y_1(k)$ are Bessel functions of the second kind and of the zero and first order. The real part of this equation is

$$C = \sqrt{b}\, v\beta_0\left(\sqrt{(X + Y)} \cos \omega t - \sqrt{(X - Y)} \sin \omega t\right)$$

where

$$X = \left(\frac{2}{\pi k}\right)^2 \frac{1}{\left(J_0 - Y_1\right)^2 + \left(J_1 + Y_0\right)^2}$$

$$Y = \left(\frac{2}{\pi k}\right)^2 \frac{\left(J_0 - Y_1\right)^2 - \left(J_1 + Y_0\right)^2}{\left[\left(J_0 - Y_1\right)^2 + \left(J_1 + Y_0\right)^2\right]^2}$$

and finally

$$s = \pi\rho b v^2 {\beta_0}^2 \left(X + Y \cos 2\omega t - Z \sin 2\omega t\right) \tag{19}$$

where

$$Z = \sqrt{X^2 - Y^2}$$

DYNAMICS OF AN AIRFOIL IN AN OSCILLATING STREAM

The problem treated herein is that of an airfoil executing torsional vibrations in an oscillating stream. Because of assumptions (c) and (d) of the Introduction - namely, that the oscillations are small and that the wake is plane - the aerodynamic moments due to partial motions of the fluid and those due to motions of the airfoil can be treated independently.

The pitching moment acting on an airfoil undergoing angular oscillations in a uniform stream is (reference 5)

$$M_a = -\pi\rho b^2\left[\left(\frac{1}{2} - a\right)vb\dot{\alpha} + b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha}\right] + 2\pi\rho vb^2\left(a + \frac{1}{2}\right)C(k)\left[v\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right] \tag{20}$$

Thus, the total aerodynamic moment acting on an oscillating airfoil in an oscillating stream is the sum of equations (15) and (20). The equation of motion is obtained by expressing the equilibrium of the aerodynamic moment, the moment of inertia, and the mechanical restoring moment. Thus, if structural damping is neglected, the equation of motion is

$$\text{Aerodynamic moment} = I_a\ddot{\alpha} + R_a\alpha \tag{21}$$

where

I_a moment of inertia about $x = a$

R_a torsional stiffness

Combining equations (15), (20), and (21) gives

$$\left[r_a^2 + \kappa\left(\frac{1}{8} + a^2\right)\right]\ddot{\alpha} + \frac{\kappa v}{b}\left(\frac{1}{2} - a\right)\left[1 - 2\left(a + \frac{1}{2}\right)C(k)\right]\dot{\alpha} + \left[\frac{R_a}{mb^2} - \frac{2\kappa}{b^2}v^2\left(a + \frac{1}{2}\right)C(k)\right]\alpha$$

$$= -\frac{2\kappa v^2}{b^2}\left(a + \frac{1}{2}\right)\beta_0\left[C(k)\left(J_0 - iJ_1\right) + iJ_1\right]e^{i\omega t} \tag{22}$$

where

m mass per unit length of wing

r_a radius of gyration divided by b $\left(\sqrt{\frac{I_a}{mb^2}}\right)$

κ ratio of mass of cylinder of air of diameter equal to chord of wing to mass of wing $\left(\frac{\pi\rho b^2}{m}\right)$

The differential equation (22) is an equation for forced vibrations, the solution to which is found by letting

$$\alpha = \alpha_0 e^{i(\omega t - \psi)} \tag{23}$$

where

α_0 amplitude of angular oscillations of the wing

ψ phase factor

The ratio of the amplitude of airfoil oscillations to the amplitude of stream oscillations will be called the response of the airfoil. If the right-hand side of equation (23) is substituted into equation (22), after somewhat lengthy but straightforward calculations the square of the response of the airfoil is found to be

$$\left(\frac{\alpha_0}{\beta_0}\right)^2 = \frac{\left(a + \frac{1}{2}\right)^2\left[\left(FJ_0 + GJ_1\right)^2 + \left(J_1 + GJ_0 - FJ_1\right)^2\right]}{\left[-\left(a + \frac{1}{2}\right)F + k\left(\frac{1}{4} - a^2\right)G - \frac{k^2}{2}\left(\frac{r_a^2}{\kappa} + \frac{1}{8} + a^2\right) + \frac{R_a}{2\kappa M v^2}\right]^2 + \left[\frac{k}{2}\left(\frac{1}{2} - a\right) - k\left(\frac{1}{4} - a^2\right)F - \left(a + \frac{1}{2}\right)G\right]^2} \tag{24}$$

where $F(k) + iG(k) = C(k)$, Theodorsen's C-function.

DISCUSSION OF RESULTS

Propulsive force.- It has been shown (see equation (19)) that the propulsive force acting on a fixed airfoil at zero angle of attack in an oscillating stream is

$$s = \pi\rho b v^2 \beta_0^2 (X + Y \cos 2\omega t - Z \sin 2\omega t)$$

Since $2\pi\rho b v^2 \beta_0$ is the stationary value of the lift on an airfoil at angle of attack β_0, let

$$L_0 = 2\pi\rho b v^2$$

Then

$$s = \frac{L_0 \beta_0}{2} (X + Y \cos 2\omega t - Z \sin 2\omega t) \tag{25}$$

and the average value of s is

$$\bar{s} = \frac{L_0 \beta_0}{2} X$$

In figures 3 and 4 curves are presented that show the variation with wave lengths of stream oscillations of the coefficients X, Y, and Z appearing in the equation for the propulsive force. For very low frequencies - that is, for long wave lengths of stream oscillations - X and Y approach the value 1, and Z becomes zero. Thus, as $k \rightarrow 0$, equation (25) becomes

$$s \approx \frac{L_0 \beta_0}{2} (1 + \cos 2\omega t) \tag{26}$$

This result is exactly that which is to be expected from quasi-stationary considerations in which the lift is assumed to be instantaneously that value prescribed by the geometrical angle of attack; that is, the shed vorticity produced by variations in angle of attack is assumed to appear instantaneously at a point infinitely distant from the airfoil. Thus (see fig. 5) the perpendicular force L on an airfoil at angle of attack β with respect to the stream is

$$L = 2\pi\rho v^2 b\beta$$

and because from well-known considerations of two-dimensional flows there can be no induced drag, the propulsive force must be

$$s = L\beta = 2\pi\rho b v^2 \beta^2$$

and if $\beta = \beta_0 \cos \omega t$,

$$s = 2\pi\rho b v^2 \beta_0{}^2 \cos^2 \omega t$$

$$= L_0\beta_0\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega t\right)$$

which is the same as equation (26).

The reason for plotting the coefficients Y and Z against k as well as against $\frac{1}{k}$, where $\frac{2\pi b}{k}$ is the wave length of stream oscillations, is that in the neighborhood of $\frac{1}{k} = 0$ the values of Y and Z fluctuate infinitely many times. This behavior is brought out in figure 4 where it may be seen that between $k = 1$ $\left(\frac{1}{k} = 1\right)$ and $k = \infty$ $\left(\frac{1}{k} = 0\right)$ the curves oscillate about zero with decreasing amplitude.

<u>Forced vibrations</u>.- An analysis is made of the response (see equation (24)) of an airfoil in an oscillating stream with particular emphasis on the parameters a and R_a, these parameters having qualitative as well as quantitative effects on the values of the response. The stiffness in torsion R_a is related to the natural frequency ν' for zero stream velocity by the equation

$$\nu' = \frac{\omega'}{2\pi}$$

$$= \frac{1}{2\pi}\sqrt{\frac{R_a}{I_a + \pi\rho b^4\left(\frac{1}{8} + a^2\right)}} \tag{27}$$

where $I_a + \pi\rho b^4\left(\frac{1}{8} + a^2\right)$ is the total moment of inertia of the airfoil. From equation (27),

$$\frac{R_a}{2\kappa m v^2} = \frac{k'^2}{2}\left[\frac{r_a{}^2}{\kappa} + \left(\frac{1}{8} + a^2\right)\right]$$

where $k' = \frac{\omega' b}{v}$.

In figure 6 are shown response curves for an airfoil hinged at the leading edge ($a = -1.0$) and having various natural frequencies of oscillation. The following values of the parameters κ and $r_a{}^2$ were chosen as being within the practical range of application: $\kappa = 0.0653$; $r_a{}^2 = \frac{4}{3}$ (flat-plate mass distribution). The response for any natural frequency is never very large $\left(\text{at most about } \frac{\alpha_0}{\beta_0} = 2.5\right)$, and the response decreases with increasing stiffness. The value of the stream frequency at which maximum response occurs is seen to correspond more closely to ν' as the natural frequency increases. Even for a freely hinged airfoil ($k' = 0$) a sort of resonance frequency exists. (See fig. 7.) Thus the use of such a device for measuring angular variations in an oscillating stream would be valid only in the range of very low reduced frequency (long wave lengths) in which the response approaches unity.

Somewhat different phenomena occur when the hinge point is behind the quarter-chord point of the airfoil. In general it may be stated that the response is greater. In particular a critical stiffness exists below which the airfoil is in unstable equilibrium. The condition for divergent motions of the airfoil is that the coefficient of α in equation (22) be less than or equal to 0 for $k = 0$. The critical stiffness, therefore, is defined by

$$\frac{R_a}{2\kappa mv^2} = a + \frac{1}{2}$$

In figure 8 are presented some results for the hinge placed near the center of the airfoil. The values of the parameters a, κ, and r_a^2 chosen were: $a = -0.1$, $\kappa = 0.0653$, and $r_a^2 = \frac{2.062}{6}$. The reduced critical frequency for this case is $k' = 0.385$. For stiffness values somewhat higher than the critical, the response is not unduly large and again, as when the hinge was at the leading edge, the maximum response with large values of the stiffness occurs at a stream frequency close to the frequency ν'.

CONCLUSIONS

The velocity potential, lift force, moment, and propulsive force on a two-dimensional airfoil in a stream of periodically varying angle of attack has been derived on the basis of nonstationary incompressible potential-flow theory which includes the effect of the continuous sheet of vortices shed from the trailing edge. Application of these results was made in an analysis of the variation with frequency of the propulsive forces on an airfoil in an oscillating stream and in an analysis of the problem of forced vibration of an airfoil in an oscillating stream with consideration of the stiffness of the airfoil and the position of its torsion axis. The following conclusions were indicated:

1. The value of the propulsive force acting on an airfoil in an oscillating stream is sufficiently large to be of practical importance.

2. The amplitude of vibration of an airfoil in an oscillating stream is critically dependent on the stiffness of the airfoil and the position of its torsion axis. In general, amplitudes of vibration are smaller when the torsion axis is ahead of the quarter-chord point and larger when the torsion axis is behind the quarter-chord point. Because of the nonuniform response of

a freely hinged airfoil the use of such a device for the measurement of angular variations in an oscillating stream would be restricted to the range of very low frequency in which the response approaches unity.

Langley Memorial Aeronautical Laboratory
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APPENDIX A

NONCIRCULATORY VELOCITY POTENTIAL FOR AN AIRFOIL IN A SINUSOIDAL GUST

The problem of finding the velocity potential for an airfoil having a certain normal velocity distribution is solved by the method indicated by von Kármán and Burgers (reference 6, p. 44).

Represent the wing by a circle (fig. 9). Place a source of strength 2ϵ at the point (x_1, y_1) on the circle and a sink of strength -2ϵ at $(x_1, -y_1)$. The velocity potential of this source-sink pair in the notation of the present paper is given by (reference 5)

$$\Delta\varphi = \frac{\epsilon}{2\pi} \log \frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_1)^2 + (y + y_1)^2}$$

The transformation of the circle to its diameter is

$$y = \sqrt{1 - x^2}, \quad x = x$$

For the airfoil in a sinusoidal gust

$$\epsilon = v\beta_0 e^{i(\omega t - kx_1)}$$

Thus

$$\Delta\varphi = \frac{v\beta_0}{2\pi} e^{i\omega t} e^{-ikx_1} \log \frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_1)^2 + (y + y_1)^2}$$

$$\varphi = b \int_{-1}^{1} \Delta\varphi \, dx_1$$

$$= \frac{v\beta_0 b}{2\pi} e^{i\omega t} \int_{-1}^{1} e^{-ikx_1} \log \frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_1)^2 + (y + y_1)^2} \, dx_1$$

Integration by parts leads to

$$\varphi = -\frac{v\beta_0 b}{i\pi k} \sqrt{1 - x^2} \, e^{i\omega t} \int_{-1}^{1} \frac{e^{-ikx_1} \, dx_1}{\sqrt{1 - x_1^2} \, (x - x_1)} \tag{A1}$$

Let

$$\int_{-1}^{1} \frac{e^{-ikx_1} \, dx_1}{\sqrt{1 - x_1^2} \, (x - x_1)} = f(k, x)$$

Then

$$\frac{\partial f}{\partial k} = -i \int_{-1}^{1} \frac{e^{-ikx_1} (x_1 - x + x) \, dx_1}{\sqrt{1 - x_1^2} \, (x - x_1)}$$

$$= i \int_{-1}^{1} \frac{e^{-ikx_1} \, dx_1}{\sqrt{1 - x_1^2}} - ix \int_{-1}^{1} \frac{e^{-ikx_1} \, dx_1}{\sqrt{1 - x_1^2} \, (x - x_1)}$$

But, from relation (4) on page 48 of reference 7,

$$\int_{-1}^{1} \frac{e^{-ikx_1}\,dx_1}{\sqrt{1 - x_1^2}} = \pi J_0(k)$$

where $J_0(k)$ is a Bessel function of the first kind and zero order. Therefore,

$$\frac{\partial f}{\partial k} = i\pi J_0(k) - ixf \tag{A2}$$

Equation (A2) is a non-homogeneous differential equation of the first order. The homogeneous part

$$\frac{\partial f}{\partial k} = -ixf$$

has as solution

$$f = ce^{-ikx}$$

where c is an arbitrary constant with respect to k. The particular solution is obtained by the method of variation of parameters (reference 8, p. 114). Let

$$f = g(k, x)e^{-ikx} \tag{A3}$$

Then

$$\frac{\partial f}{\partial k} = \frac{\partial g}{\partial k} e^{-ikx} - ixge^{-ikx}$$

Combining this expression with equation (A2) gives

$$\frac{\partial g}{\partial k} = i\pi e^{ikx} J_0(k)$$

integration of which leads to

$$g = i\pi \int_0^k e^{iux} J_0(u)\,du \tag{A4}$$

Combining equations (A3) and (A4) gives

$$f(k;\,x) = i\pi e^{-ikx} \int_0^k e^{iux} J_0(u)\,du$$

which, when substituted into equation (A1), gives for the noncirculatory velocity potential

$$\varphi = -\frac{bv\beta_0}{k} \sqrt{1 - x^2}\, e^{i(\omega t-kx)} \int_0^k e^{iux} J_0(u)\,du \tag{A5}$$

APPENDIX B

LIFT AND MOMENT FOR NONCIRCULATORY FLOW

Lift

The total force on the airfoil is given by

$$P = b \int_{-1}^{1} \Delta p \, dx$$

where, from equation (2)

$$\Delta p = -2\rho \left(\frac{v}{b} \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} \right)$$

But

$$\varphi_{x=1} = \varphi_{x=-1} = 0$$

Therefore

$$P = -2\rho b \int_{-1}^{1} \frac{\partial \varphi}{\partial t} \, dx$$

From equation (1),

$$\frac{\partial \varphi}{\partial t} = -\frac{i\omega b v \beta_0}{k} \sqrt{1 - x^2} \, e^{i\omega t} e^{-ikx} \int_0^k e^{iux} J_0(u) \, du \tag{B1}$$

Thus

$$P = \frac{i2\rho b^2 v \beta_0 \omega}{k} \, e^{i\omega t} \int_{-1}^{1} \sqrt{1 - x^2} \, e^{-ikx} \, dx \int_0^k e^{iux} J_0(u) \, du \tag{B2}$$

Let

$$f_1(k) = \int_{-1}^{1} \sqrt{1 - x^2}\, e^{-ikx}\, dx \int_0^k e^{iux} J_0(u)\, du \tag{B3}$$

Interchanging the order of integration gives

$$f_1(k) = \int_0^k J_0(u)\, du \int_{-1}^{1} \sqrt{1 - x^2}\, e^{i(u-k)x}\, dx \tag{B4}$$

From reference 7, page 48,

$$J_\nu(z) = \frac{\left(\frac{1}{2} z\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1 - t^2)^{\nu - \frac{1}{2}}\, e^{izt} dt \tag{B5}$$

Therefore

$$J_1(u - k) = \frac{\frac{1}{2}(u - k)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}}\, e^{i(u-k)x}\, dx$$

$$= \frac{u - k}{\pi} \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}}\, e^{i(u-k)x}\, dx$$

Substituting this expression into equation (B4) gives

$$f_1(k) = \pi \int_0^k J_0(u)\, J_1(u - k)\, \frac{du}{u - k}$$

Let

$$u - k = -w$$

and

$$du = -dw$$

Then

$$f_1(k) = \pi \int_k^0 J_1(-w)\ J_0(k - w)\ \frac{dw}{w}$$

$$= -\pi \int_0^k J_1(-w)\ J_0(k - w)\ \frac{dw}{w}$$

$$= \pi \int_0^k J_1(w)\ J_0(k - w)\ \frac{dw}{w}$$

But from reference 7, page 380, for $\mu > 0$ and $\nu > -1$

$$\int_0^z J_\mu(t)\ J_\nu(z - t)\ \frac{dt}{t} = \frac{J_{\mu+\nu}(z)}{\mu} \tag{B6}$$

Therefore

$$f_1(k) = \pi J_1(k) \tag{B7}$$

and equation (B2) becomes

$$P = \frac{2\pi i\rho b^2 v\beta_0\omega}{k} J_1(k)\ e^{i\omega t} \tag{B8}$$

Moment

The noncirculatory moment about the point $x = a$ (fig. 2) is obtained from the integral

$$M_a = b^2 \int_{-1}^{1} \Delta p(x - a)\ dx$$

which, combined with equation (2), gives

$$M_a = 2\rho bv \int_{-1}^{1} \varphi\ dx + 2\rho b^2 a \int_{-1}^{1} \frac{\partial\varphi}{\partial t}\ dx - 2\rho b^2 \int_{-1}^{1} x \frac{\partial\varphi}{\partial t}\ dx$$

By use of equations (1) and (B1), this equation becomes

$$\begin{aligned} M_a = &- \frac{2\rho b^2 v^2 \beta_0}{k} e^{i\omega t} \int_{-1}^{1} \sqrt{1 - x^2}\ e^{-ikx}\ dx \int_0^k e^{iux} J_0(u)\ du \\ &- \frac{2i\rho b^3 \omega a v\beta_0}{k} e^{i\omega t} \int_{-1}^{1} \sqrt{1 - x^2}\ e^{-ikx}\ dx \int_0^k e^{iux} J_0(u)\ du \\ &+ \frac{2i\rho b^3 \omega v\beta_0}{k} e^{i\omega t} \int_{-1}^{1} x\sqrt{1 - x^2}\ e^{-ikx}\ dx \int_0^k e^{iux} J_0(u)\ du \end{aligned} \tag{B9}$$

The first two integrals on the right side of this expression are already known (see equations (B3) and (B7)). In order to obtain the third integral, let

$$f_2(k) = \int_{-1}^{1} x\sqrt{1 - x^2}\, e^{-ikx}\, dx \int_0^k e^{iux} J_0(u)\, du \tag{B10}$$

Interchange the order of integration. Then

$$f_2(k) = \int_0^k J_0(u)\, du \int_{-1}^{1} x\sqrt{1 - x^2}\, e^{i(u-k)x}\, dx \tag{B11}$$

But

$$\int_{-1}^{1} x\sqrt{1 - x^2}\, e^{i(u-k)x}\, dx = -\frac{1}{3}(1 - x^2)^{3/2} e^{i(u-k)x} \Bigg|_{-1}^{1} \tag{B12}$$

$$+ \frac{i(u - k)}{3} \int_{-1}^{1} (1 - x^2)^{3/2} e^{i(u-k)x}\, dx$$

$$= \frac{i(u - k)}{3} \int_{-1}^{1} (1 - x^2)^{3/2} e^{i(u-k)x}\, dx$$

and, from equation (B5)

$$\int_{-1}^{1} (1 - x^2)^{3/2} e^{i(u-k)x}\, dx = \frac{3\pi J_2(u - k)}{(u - k)^2} \tag{B13}$$

Combine equations (B12) and (B13) and substitute into equation (B11). Then

$$f_2(k) = i\pi \int_0^k J_0(u)\ J_2(u - k)\ \frac{du}{u - k}$$

Let

$$u - k = -w$$

and

$$du = -dw$$

Then

$$f_2(k) = i\pi \int_k^0 J_2(-w)\ J_0(k - w)\ \frac{dw}{w} = -i\pi \int_0^k J_2(-w)\ J_0(k - w)\ \frac{dw}{w}$$

$$= -i\pi \int_0^k J_2(w)\ J_0(k - w)\ \frac{dw}{w}$$

which becomes (see equation (B6))

$$f_2(k) = -\frac{i\pi}{2} J_2(k) \tag{B14}$$

The equation for the moment about $x = a$ can now be written as

$$M_a = -\frac{2\pi\rho b^2 v^2 \beta_0}{k} J_1(k)\ e^{i\omega t} - \frac{2\pi i\rho b^3 a\omega v\beta_0}{k} J_1(k) e^{i\omega t} + \frac{\pi\rho b^3 \omega v\beta_0}{k} J_2(k)\ e^{i\omega t} \tag{B15}$$

The recurrence formula from reference 7, page 17,

$$J_{n-1}(k) + J_{n+1}(k) = \frac{2n}{k} J_n(k)$$

gives for $n = 1$

$$J_2(k) = -J_0(k) + \frac{2}{k} J_1(k)$$

Substituting this expression into equation (B15) and making use of the definition $k = \frac{\omega b}{v}$ yields

$$M_a = -2\pi i \rho b^2 a v^2 \beta_0 \ J_1(k) \ e^{i\omega t} - \pi \rho b^2 v^2 \beta_0 J_0(k) \ e^{i\omega t} \tag{B16}$$

APPENDIX C

EVALUATION OF Q (EQUATION (8))

Equation (8) is

$$Q = -\frac{v\beta_0}{k} e^{i(\omega t-k)} \int_0^k e^{iu} J_0(u)\, du$$

$$= -\frac{v\beta_0}{k} e^{i\omega t} \int_0^k e^{i(u-k)} J_0(u)\, du \tag{C1}$$

But

$$\int_0^k e^{i(u-k)} J_0(u)\, du = \int_0^k \cos\,(u-k)\, J_0(u)\, du$$

$$+ i \int_0^k \sin\,(u-k)\, J_0(u)\, du$$

From reference 7, pages 380 and 381,

$$\int_0^k \cos\,(k-u)\, J_0(u)\, du = kJ_0(k)$$

$$\int_0^k \sin\,(k-u)\, J_0(u)\, du = kJ_1(k)$$

Therefore

$$\int_0^k e^{i(u-k)} J_0(u)\, du = kJ_0(k) - ikJ_1(k)$$

$$= k\left[J_0(k) - iJ_1(k)\right]$$

and

$$Q = -v\beta_0\left[J_0(k) - iJ_1(k)\right]e^{i\omega t} \tag{C2}$$

APPENDIX D

CALCULATION OF THE PROPULSIVE FORCE

From equation (1)

$$\left(\frac{\partial\varphi}{\partial x}\right)_{x=-1} = -\left(\frac{1}{\sqrt{1-x^2}}\right)_{x=-1} \frac{bv\beta_0}{k} e^{i(\omega t+k)} \int_0^k e^{-iu} J_0(u)\, du$$

$$= -\left(\frac{1}{\sqrt{1-x^2}}\right)_{x=-1} \frac{bv\beta_0}{k} e^{i\omega t} \int_0^k e^{-i(u-k)} J_0(u)\, du$$

The integral on the right side of this equation is the complex conjugate of that in appendix C. Thus

$$\int_0^k e^{-i(u-k)} J_0(u)\, du = k\overline{(J_0 - iJ_1)}$$

$$= k(J_0 + iJ_1)$$

where J_0 means $J_0(k)$, and so forth. Therefore

$$\left(\frac{\partial\varphi}{\partial x}\right)_{x=-1} = -\left(\frac{1}{\sqrt{1+x}}\right)_{x=-1} \frac{bv\beta_0}{\sqrt{2}} (J_0 + iJ_1)\, e^{i\omega t} \tag{D1}$$

Now, for the velocity due to the wake,

$$\left(\frac{\partial\varphi_\Gamma}{\partial x}\right)_{x=-1} = \left(\frac{1}{\sqrt{1-x^2}}\right)_{x=-1} \frac{b}{2\pi}\int_1^\infty \frac{\sqrt{x_0^2-1}}{x_0+1}\, U\, dx_0$$

but (see equations (7) and (8))

$$\frac{1}{2\pi}\int_{1}^{\infty}\frac{\sqrt{x_0^2 - 1}}{x_0^2 - 1}\,U\,dx_0 = Q$$

Therefore

$$\left(\frac{\partial\varphi_\Gamma}{\partial x}\right)_{x=-1} = \left(\frac{1}{\sqrt{1 - x^2}}\right)_{x=-1} bQ\,\frac{\displaystyle\int_1^{\infty}\frac{x_0 - 1}{\sqrt{x_0^2 - 1}}\,e^{-ikx_0}\,dx_0}{\displaystyle\int_1^{\infty}\frac{x_0 + 1}{\sqrt{x_0^2 - 1}}\,e^{-ikx_0}\,dx_0} \tag{D2}$$

From reference 7, page 180,

$$J_n(k) = \frac{2}{\pi}\int_0^{\infty}\sin\left(k\cosh t - \frac{1}{2}n\pi\right)\cosh nt\,dt$$

and

$$Y_n(k) = -\frac{2}{\pi}\int_0^{\infty}\cos\left(k\cosh t - \frac{1}{2}n\pi\right)\cosh nt\,dt$$

which lead to, when cosh t is replaced by x_0,

$$J_0(k) = \frac{2}{\pi}\int_1^\infty \frac{\sin k\, x_0\, dx_0}{\sqrt{x_0^2 - 1}}$$

$$J_1(k) = -\frac{2}{\pi}\int_1^\infty \frac{x_0 \cos k\, x_0\, dx_0}{\sqrt{x_0^2 - 1}}$$

$$Y_0(k) = -\frac{2}{\pi}\int_1^\infty \frac{\cos k\, x_0\, dx_0}{\sqrt{x_0^2 - 1}}$$

$$Y_1(k) = -\frac{2}{\pi}\int_1^\infty \frac{x_0 \sin k\, x_0\, dx_0}{\sqrt{x_0^2 - 1}}$$

Equation (D2) becomes now

$$\left(\frac{\partial\varphi_\Gamma}{\partial x}\right)_{x=-1} = \left(\frac{1}{\sqrt{1 + x}}\right)_{x=-1} \frac{b}{\sqrt{2}}\, Q\, \frac{(J_1 - Y_0) - i(J_0 + Y_1)}{(J_1 + Y_0) + i(J_0 - Y_1)}$$

and after substitution of the value of Q from equation (C2)

$$\left(\frac{\partial\varphi_\Gamma}{\partial x}\right)_{x=-1} = -\left(\frac{1}{\sqrt{1+x}}\right)_{x=-1} \frac{bv\beta_0}{\sqrt{2}} (J_0 - iJ_1) \left[\frac{(J_1 - Y_0) - i(J_0 + Y_1)}{(J_1 + Y_0) + i(J_0 - Y_1)}\right] e^{i\omega t} \tag{D3}$$

Combining equations (D1) and (D3) with (see equation (18))

$$C = \left[\sqrt{\frac{1+x}{b}}\left(\frac{\partial\varphi}{x} + \frac{\partial\varphi_\Gamma}{\partial x}\right)\right]_{x=-1}$$

gives

$$C = -\sqrt{\frac{b}{2}}\, v\beta_0 e^{i\omega t}\left[(J_0 + iJ_1) + (J_0 - iJ_1)\frac{(J_1 - Y_0) - i(J_0 + Y_1)}{(J_1 + Y_0) + i(J_0 - Y_1)}\right] \tag{D4}$$

$$= -\sqrt{\frac{b}{2}}\, v\beta_0(\cos\omega t + i\sin\omega t)\left\{J_0\left[1 + \frac{J_1^2 - J_0^2 + Y_1^2 - Y_0^2}{(J_1 + Y_0)^2 + (J_0 - Y_1)^2}\right]\right.$$

$$- \frac{2J_1(J_0J_1 + Y_0Y_1)}{(J_1 + Y_0)^2 + (J_0 - Y_1)^2} + iJ_1\left[1 - \frac{J_1^2 - J_0^2 + Y_1^2 - Y_0^2}{(J_1 + Y_0)^2 + (J_0 - Y_1)^2}\right]$$

$$\left. - i\frac{2J_0(J_0J_1 + Y_0Y_1)}{(J_1 + Y_0)^2 + (J_0 - Y_1)^2}\right\}$$

the real part of which is

$$C = \sqrt{b}\, v\beta_0\left[\sqrt{(X + Y)}\cos\omega t - \sqrt{(X - Y)}\sin\omega t\right] \tag{D5}$$

where

$$X = \frac{(J_0Y_1 - J_1Y_0)^2{}_0}{(J_0 - Y_1)^2 + (J_1 + Y_0)^2}$$

$$= \left(\frac{2}{\pi k}\right)^2 \frac{1}{(J_0 - Y_1)^2 + (J_1 + Y_0)^2}$$

and

$$Y = \frac{(J_0Y_1 - J_1Y_0)^2 \left[(J_0 - Y_1)^2 - (J_1 + Y_0)^2\right]}{\left[(J_0 - Y_1)^2 + (J_1 + Y_0)^2\right]^2}$$

$$= \left(\frac{2}{\pi k}\right)^2 \frac{(J_0 - Y_1)^2 - (J_1 + Y_0)^2}{\left[(J_0 - Y_1)^2 + (J_1 + Y_0)^2\right]^2}$$

and use has been made of the formula (reference 7, p. 77)

$$J_0Y_1 - J_1Y_0 = -\frac{2}{\pi k}$$

Substituting equation (D5) into (see equation (17))

$$s = \pi\rho C^2$$

gives

$$s = \pi\rho b v^2 \beta_0{}^2 \left[(X + Y) \cos^2\omega t + (X - Y) \sin^2\omega t \right.$$

$$\left. -2\sqrt{X^2 - Y^2} \sin \omega t \cos \omega t \right]$$

$$= \pi\rho b v^2 \beta_0{}^2 \left(X + Y \cos 2\omega t - \sqrt{X^2 - Y^2} \sin 2\omega t \right)$$

Letting $Z = \sqrt{X^2 - Y^2}$ gives finally

$$s = \pi\rho b v^2 \beta_0{}^2 (X + Y \cos 2\omega t - Z \sin 2\omega t) \tag{D6}$$

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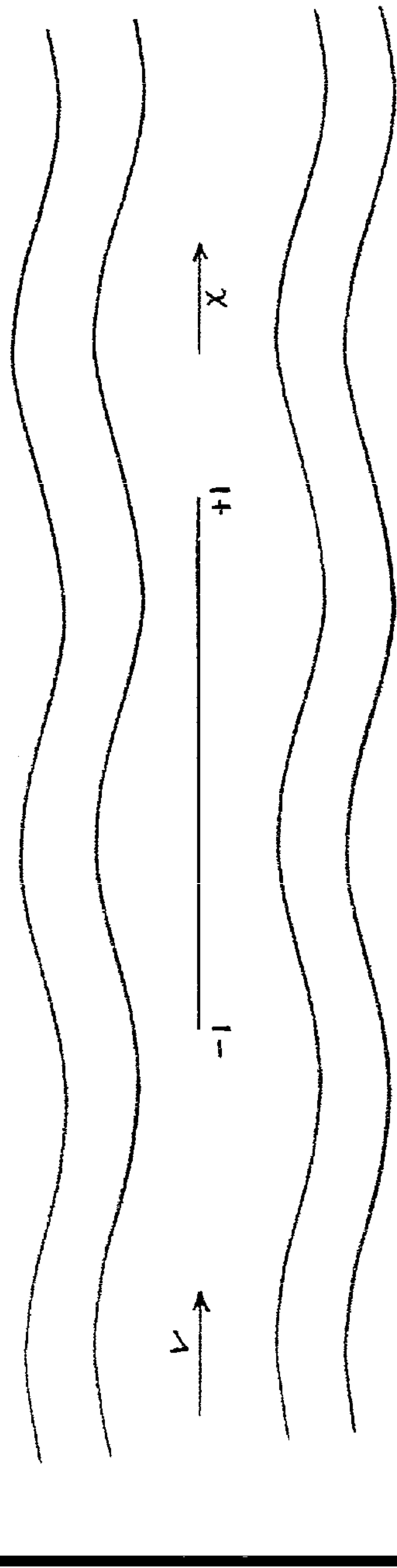

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Figure 1.- Schematic drawing of an airfoil in an oscillating stream. (Horizontal coordinate nondimensional with respect to the half chord b.)

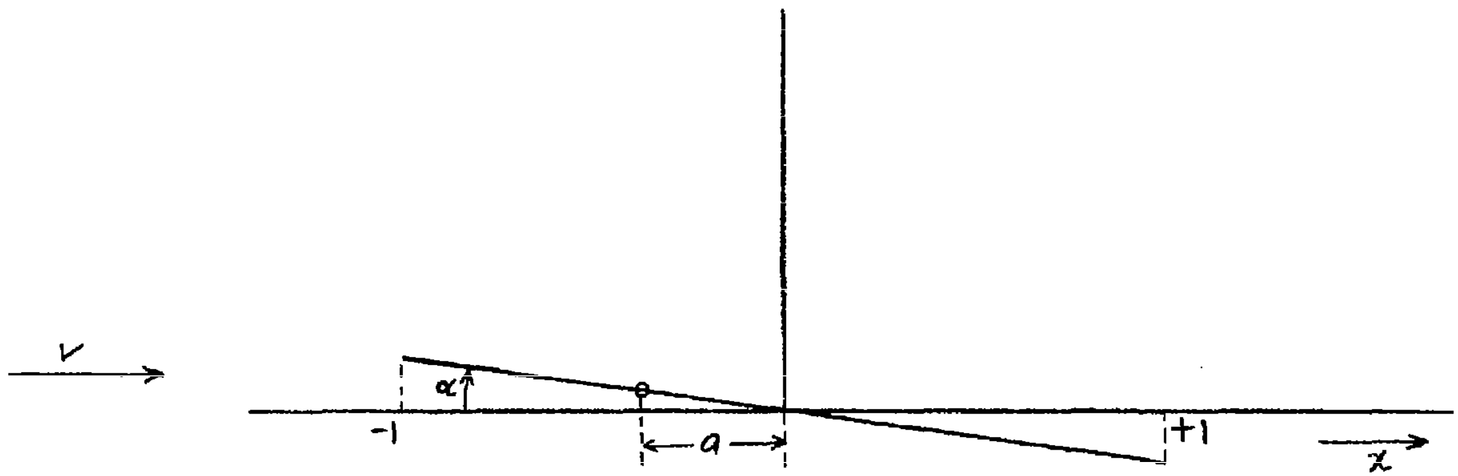

Figure 2.– Diagram showing notation used in the case of an oscillating airfoil. (Horizontal coordinate nondimensional with respect to b.)

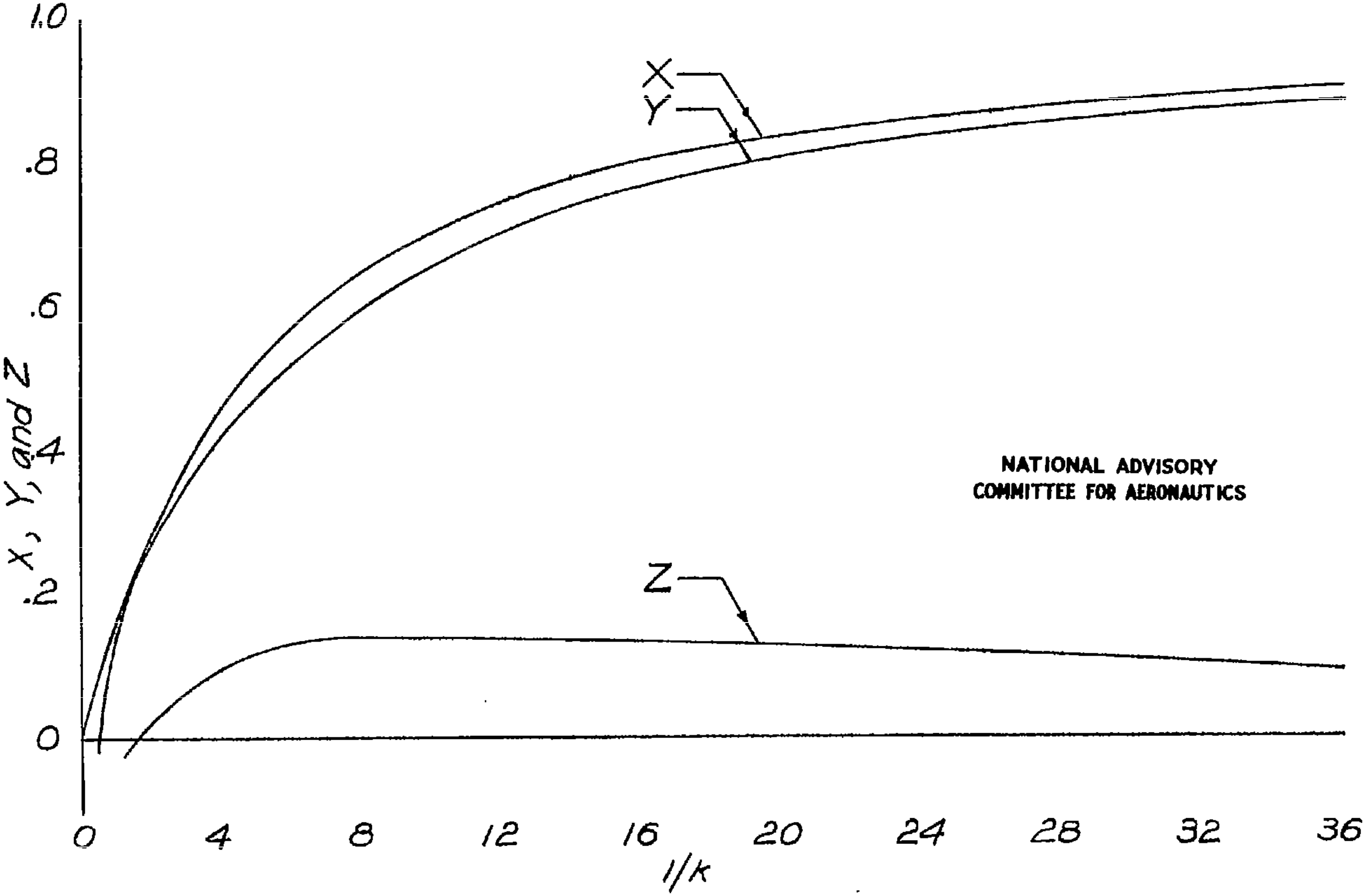

Figure 3.- Variation with wave length of stream oscillations of the coefficients X, Y, and Z appearing in the equation for the propulsive force.

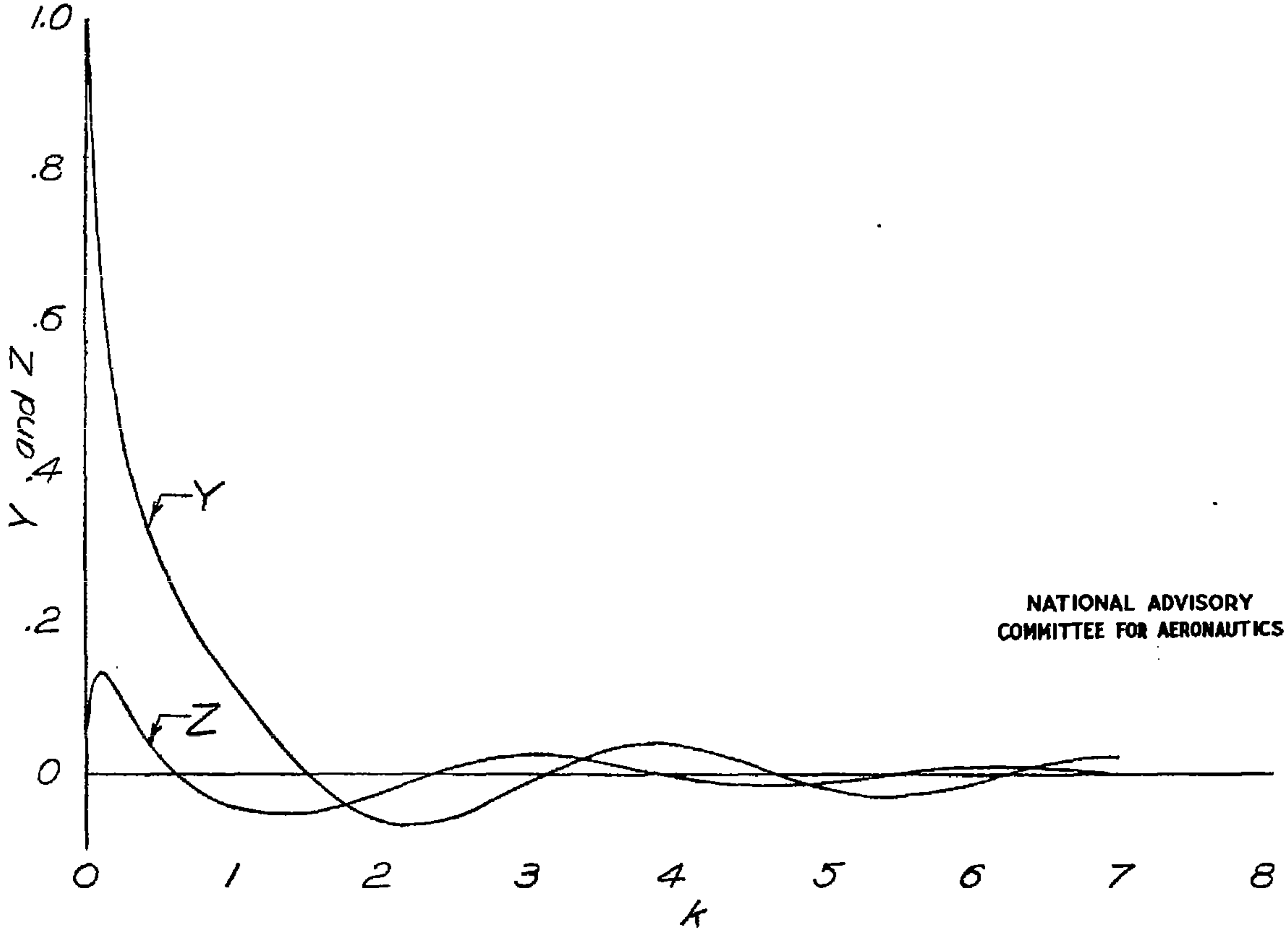

Figure 4.– Variation with reduced frequency of stream oscillations of the coefficients Y and Z.

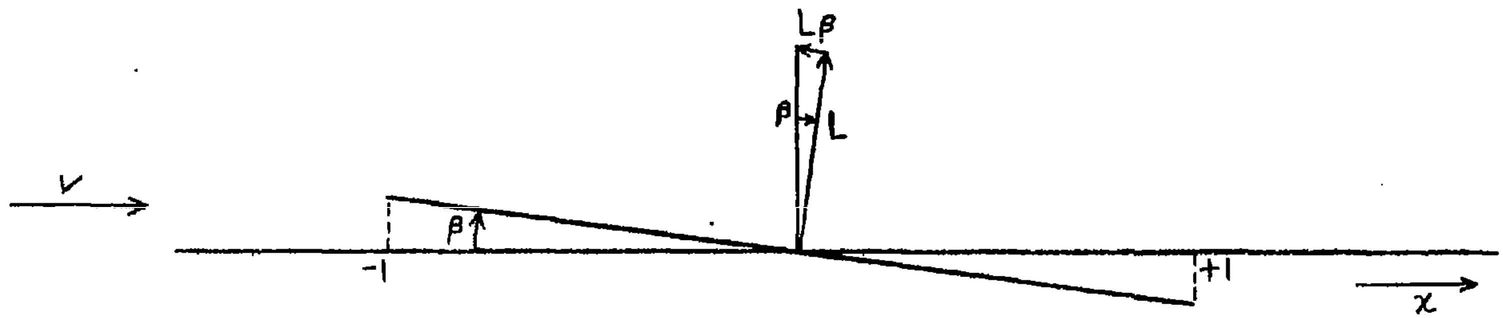

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Figure 5.— Diagram showing notation used in quasi-stationary derivation of the propulsive force on an oscillating airfoil.

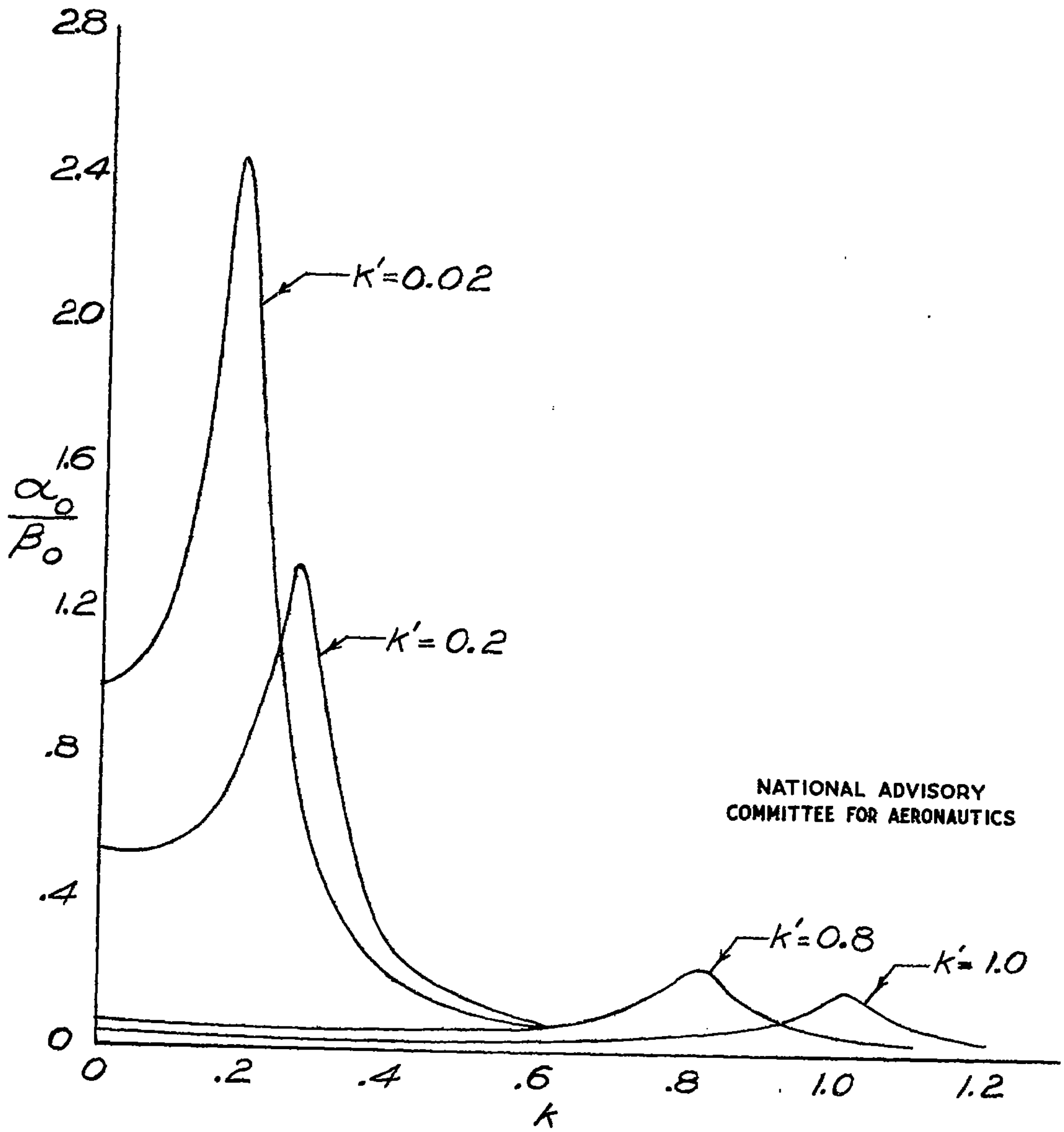

Figure 6.- Variation with reduced frequency of stream oscillations of the response of an airfoil hinged at its leading edge.

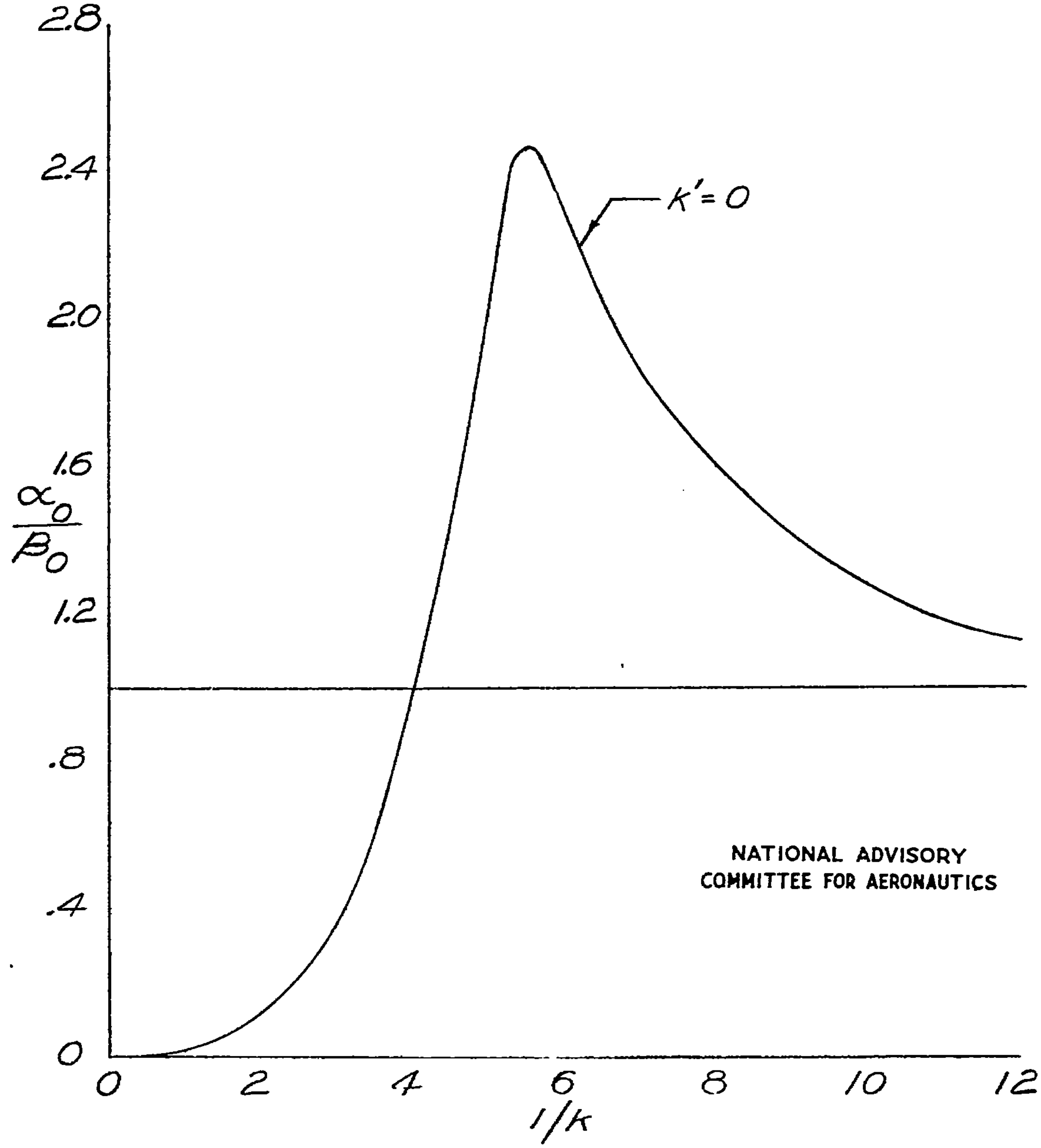

Figure 7.- Variation with wave length of stream oscillations of the response of an airfoil freely hinged at its leading edge.

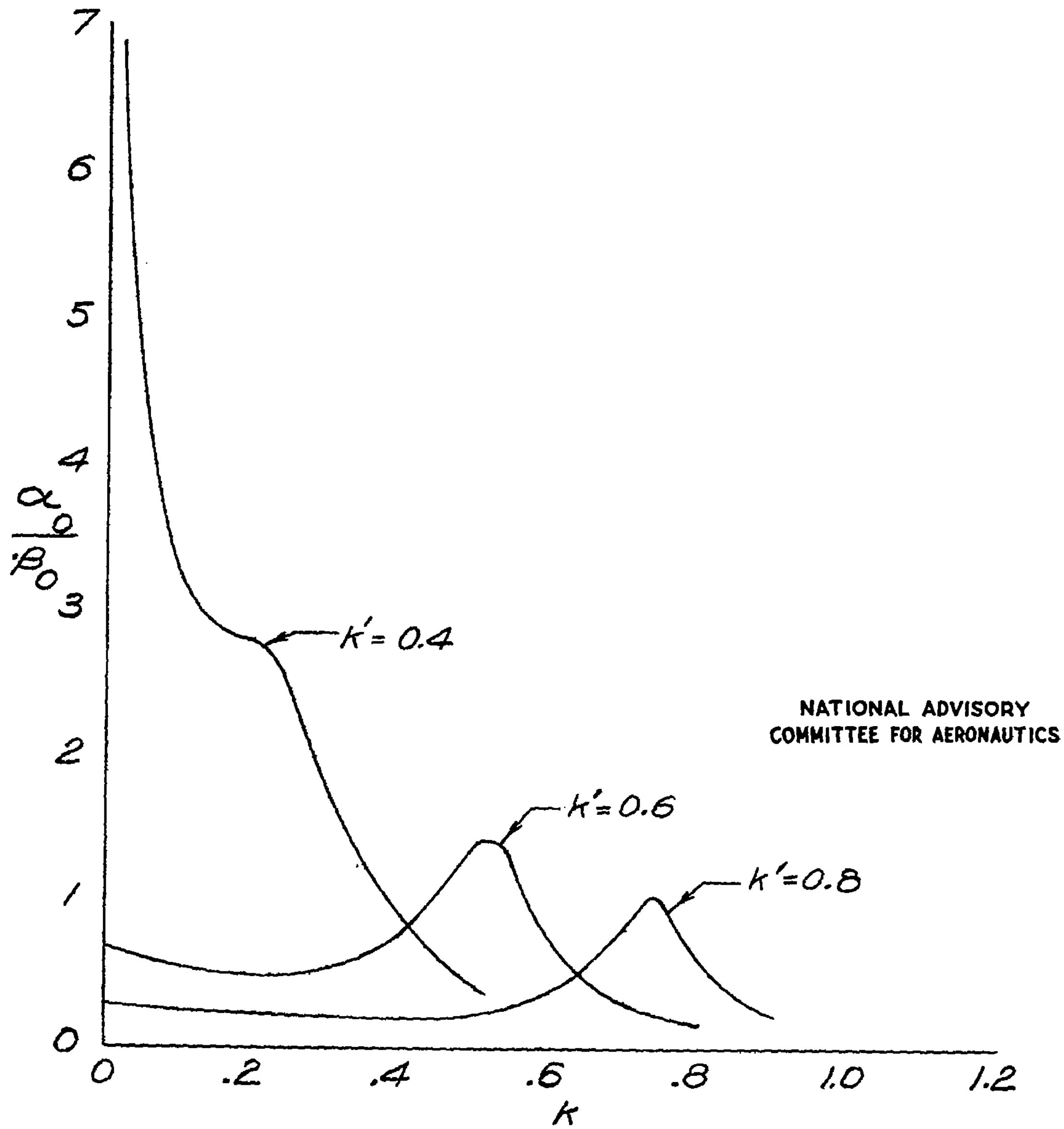

Figure 8.- Variation with reduced frequency of stream oscillations of the response of an airfoil hinged near its center.

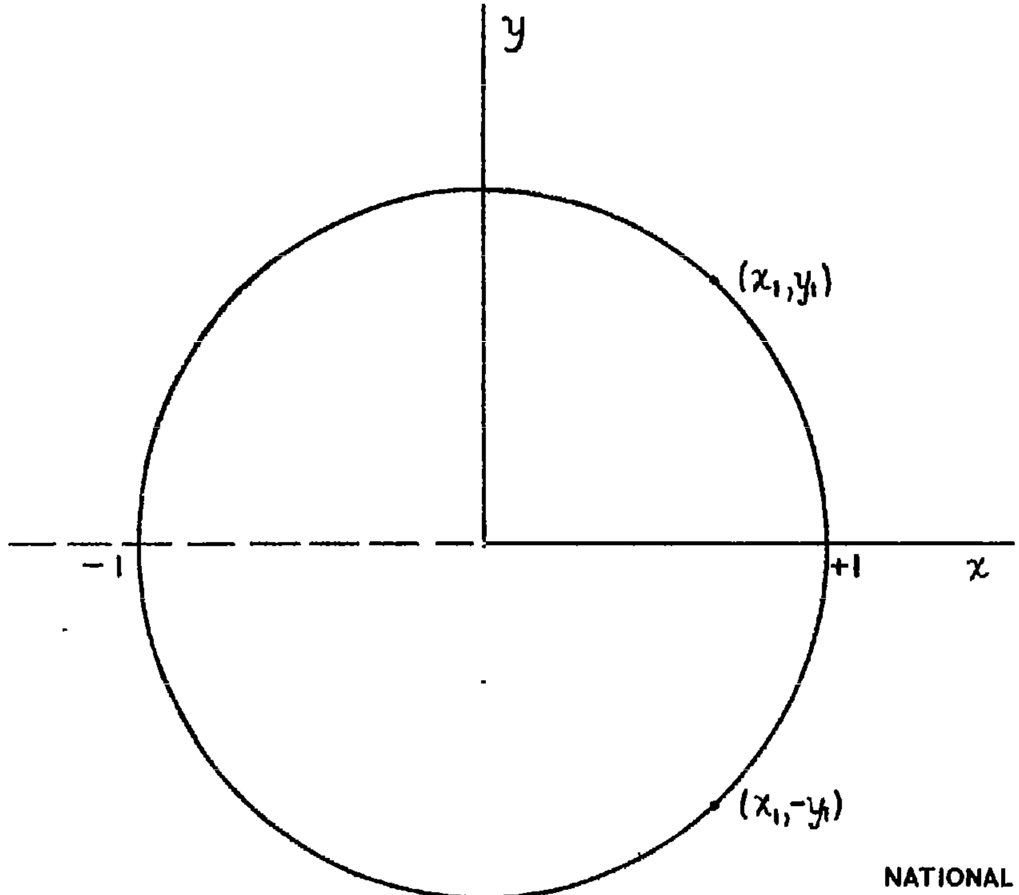

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Figure 9.– Conformal representation of the wing profile by a circle. (Linear quantities nondimensional with respect to b.)